School of Mechatronic Systems Engineering
Simon Fraser University
MSE483/782 Midterm Exam

February 22, 2018 (Duration: 2 hours)
Please read the following before signing your name

- The exam is closed-book. A 2-page formula sheet is permitted but must not contain any solved problems. The formula sheet has to be returned with the questions.
- Questions have an equal weight of 20% each. **Please clearly specify any assumptions you make and write legibly. You may lose marks if your work is not clear.**

Name: Student I.D. Number:

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1) For the following system obtain an approximate output y when a step input u with a small amplitude of ~~0.01~~ ϵ is applied to the system. Assume that the system is resting at its equilibrium point before the step input is applied.

$$\begin{aligned}
\dot{x}_1 &= (-\alpha_1 + sin(x_2))x_1 + x_2 sin(x_2) + u \\
\dot{x}_2 &= x_1 sin(x_1) + x_2(-\beta_2 + sin(x_2)) + u \\
y &= sin(x_1) + u cos(x_2)
\end{aligned}$$

Equilibrium $\longrightarrow$ $\dot{x}_1 = 0$, $\dot{x}_2 = 0$, $u = 0$ $\longrightarrow$ $x_1, x_2 = 0$ $\rightarrow$ 5

Linearization about $x_1, x_2 = 0 \longrightarrow$

$$\begin{aligned}
\dot{\tilde{x}}_1 &= -\alpha_1 \tilde{x}_1 + \tilde{u} \\
\dot{\tilde{x}}_2 &= -\alpha_2 \tilde{x}_2 + \tilde{u} \\
\tilde{y} &= \tilde{x}_1 + \tilde{u}
\end{aligned}$$

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$$\therefore \begin{cases} \dot{\tilde{x}} = \begin{bmatrix} -\alpha_1 & 0 \\ 0 & -\beta_2 \end{bmatrix} \tilde{x}_2 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tilde{u} \\ \tilde{y} = [1 \ \ 0]\, \tilde{x}_1 + 1\, \tilde{u} \end{cases}$$

$$\frac{\tilde{Y}}{\tilde{U}} = C(sI - A)^{-1}B + D \Longrightarrow H(s) = [1 \ \ 0] \begin{bmatrix} s+\alpha_1 & 0 \\ 0 & s+\beta_2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1$$

$$= [1 \ \ 0] \begin{bmatrix} \frac{1}{s+\alpha_1} & 0 \\ 0 & \frac{1}{s+\beta_2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1$$

$$= [1 \ \ 0] \begin{bmatrix} \frac{1}{s+\alpha} \\ \frac{1}{s+\beta} \end{bmatrix} + 1$$

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$$H(s) = \frac{1}{s+\alpha_1} + 1$$

$$\therefore \tilde{Y} = \frac{\epsilon}{s}\left(\frac{1}{s+\alpha} + 1\right) = \frac{\epsilon}{s(s+\alpha)} + \frac{\epsilon}{s} = \frac{\epsilon(s+\alpha-s)}{\alpha\, s(s+\alpha)} + \frac{\epsilon}{s}$$

$$= \frac{\epsilon}{\alpha s} - \frac{\epsilon}{\alpha(s+\alpha)} + \frac{\epsilon}{s} \Longrightarrow y(t) = \left[\epsilon\left(1+\frac{1}{\alpha}\right) - \frac{\epsilon}{\alpha} e^{-\alpha t}\right] 1(t)$$

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2) Obtain the diagonal canonical form representation for the system given by $\ddot{x} + 3\dot{x} + 2x = u$.

Let $\begin{cases} x_1 = x \\ x_2 = \dot{x} \end{cases} \longrightarrow \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2x_1 - 3x_2 + u \end{aligned} \longrightarrow \dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$

<u>Eigenvalues</u>: $\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{bmatrix}\right) = \lambda(\lambda+3) + 2 = \lambda^2 + 3\lambda + 2$

$\lambda^2 + 3\lambda + 2 = 0 \longrightarrow \lambda_1 = -1 \quad \lambda_2 = -2$

$Av_1 = \lambda_1 v_1 \longrightarrow \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = -\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} \longrightarrow \begin{aligned} v_{12} &= -v_{11} \\ -2v_{11} - 3v_{12} &= -v_{12} \end{aligned} \rightarrow -2v_{11} = 2v_{12}$

$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}\alpha$

$Av_2 = \lambda_2 v_2 \longrightarrow \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = -2\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} \longrightarrow \begin{aligned} v_{22} &= -2v_{21} \\ -2v_{21} - 3v_{22} &= -2v_{22} \end{aligned}$

$\longrightarrow \begin{cases} v_{22} = -2v_{21} \\ v_{22} = -2v_{21} \end{cases} \longrightarrow v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}\beta$

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$\therefore T_{DCF} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \longrightarrow x = T_{DCF} z \longrightarrow T_{DCF}\dot{z} = AT_{DCF}z + Bu$

$\therefore \dot{z} = T_{DCF}^{-1} A T_{DCF} z + T_{DCF}^{-1} B u$

$T_{DCF}^{-1} = \frac{1}{-2+1}\begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix}^T = \begin{bmatrix} +2 & +1 \\ -1 & -1 \end{bmatrix} \Longrightarrow$

$A_{diag} = T_{DCF}^{-1} A T_{DCF} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 1 & 4 \end{bmatrix}$

$= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$

$B_{diag} = T_{DCF}^{-1} B = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\therefore \dot{z} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} z + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u$

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3) For the system given by $\dot{x} = Ax$ the following state trajectories are obtained in response to two different initial conditions of the state vector x:

$$\begin{bmatrix} e^{-4t} + 2e^{2t} \\ -4e^{-4t} + 4e^{2t} \end{bmatrix}, \quad \begin{bmatrix} -e^{-4t} + e^{2t} \\ 4e^{-4t} + 2e^{2t} \end{bmatrix}. \tag{1}$$

Obtain the initial conditions, state transition matrix, and matrix A.

$$x_1 = \begin{bmatrix} e^{-4t} + 2e^{2t} \\ -4e^{-4t} + 4e^{2t} \end{bmatrix} \longrightarrow x_1(0) = \begin{bmatrix} 1+2 \\ -4+4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} -e^{-4t} + e^{2t} \\ 4e^{-4t} + 2e^{2t} \end{bmatrix} \longrightarrow x_2(0) = \begin{bmatrix} -1+1 \\ 4+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

$$x(t) = e^{At} x(0) \implies \left.\begin{aligned} x_1(t) &= e^{At} x_1(0) \\ x_2(t) &= e^{At} x_2(0) \end{aligned}\right\} \rightarrow 5$$

We can find e^{At} by forming the following equation:

$$\begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix} = e^{At} \begin{bmatrix} x_1(0) & x_2(0) \end{bmatrix}$$

$$\begin{bmatrix} e^{-4t} + 2e^{2t} & -e^{-4t} + e^{2t} \\ -4e^{-4t} + 4e^{2t} & 4e^{-4t} + 2e^{2t} \end{bmatrix} = e^{At} \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}$$

$$\therefore \; e^{At} = \begin{bmatrix} '' & '' \\ '' & '' \end{bmatrix} \times \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}^{-1} = \begin{bmatrix} '' & '' \\ '' & '' \end{bmatrix} \begin{bmatrix} 1/3 & 0 \\ 0 & 1/6 \end{bmatrix}$$

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$$= \begin{bmatrix} \frac{1}{3}(e^{-4t} + 2e^{2t}) & +\frac{1}{6}(-e^{-4t} + e^{2t}) \\ \frac{1}{3}(-4e^{-4t} + 4e^{2t}) & \frac{1}{6}(4e^{-4t} + 2e^{2t}) \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 2e^{-4t} + 4e^{2t} & -e^{-4t} + e^{2t} \\ -8e^{-4t} + 8e^{2t} & 4e^{-4t} + 2e^{2t} \end{bmatrix}$$

To find A, differentiate e^{At} wrt t and set $t = 0$, i.e.

$$A = \frac{d}{dt}\left(e^{At}\right)\Big|_{t=0} = A e^{At}\Big|_{t=0} = \frac{1}{6} \begin{bmatrix} -8e^{-4t} + 8e^{2t} & 4e^{-4t} + 2e^{2t} \\ 32e^{-4t} + 16e^{2t} & -16e^{-4t} + 4e^{2t} \end{bmatrix}_{t=0}$$

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$$= \frac{1}{6} \begin{bmatrix} 0 & 6 \\ 48 & -12 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix}$$

4) Consider the linear time-invariant system

$$\dot{x} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t)$$
$$y = [1 \;\; 0]x.$$

Determine if an input u exists such that the system states can be steered from an arbitrary initial condition(s) to zero in 1 second. Obtain such input and initial condition(s) if they exist.

$P = [B \;\; AB] = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \longrightarrow \det(P) = 0 \longrightarrow$ Uncontrollable!

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$\dot{x}_1 = -2x_1 + u \longrightarrow x_1 = x_{10} e^{-2t} + \int_0^t e^{-2(t-\tau)} u(\tau)\, d\tau$

$\dot{x}_2 = -2x_2 + 2u \longrightarrow x_2 = x_{20} e^{-2t} + \int_0^t e^{-2(t-\tau)}\, 2\, u(\tau)\, d\tau$

Final state at $t=1 \longrightarrow x_1(1) = x_2(1) = 0$

$\therefore \; 0 = x_{10} e^{-2} + \int_0^{1} e^{-2(1-\tau)} u(\tau)\, d\tau \Rightarrow 0 = x_{10}\cancel{e^{-2}} + \cancel{e^{-2}} \int_0^1 e^{2\tau} u(\tau)\, d\tau$

$0 = x_{20} e^{-2} + \int_0^1 e^{-2(1-\tau)}\, 2\, u(\tau)\, d\tau \Rightarrow 0 = x_{20}\cancel{e^{-2}} + \cancel{e^{-2}} \int_0^1 e^{2\tau}\, 2u(\tau)\, d\tau$

$\therefore \; x_{10} = -\int_0^1 e^{2\tau} u(\tau)\, d\tau$

$x_{20} = -2 \int_0^1 e^{2\tau} u(\tau)\, d\tau \implies \dfrac{x_{10}}{x_{20}} = \dfrac{1}{2}$ if such u exists

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Initial conditions

Input that can steer x_{10} to zero in 1 sec:

$\dot{x}_1 = -2x_1 + u \longrightarrow u = -B^T e^{A^T(t_0 - t)} W^{-1}(t_0, t_f)\, x_0$

where $W(t_0, t_f) = \int_{t_0}^{t_f} e^{A(t_0-\tau)} B B^T e^{A^T(t_0-\tau)}\, d\tau$

Take $A = -2,\; B = 1,\; t_0 = 0,\; t_f = 1$

$\Rightarrow W(t_0, t_f) = \int_0^1 e^{-2(0-\tau)}\, 1 \times 1\, e^{-2(0-\tau)}\, d\tau = \int_0^1 e^{4\tau}\, d\tau = \frac{1}{4}\left[e^{4\tau}\right]_0^1$

$= \frac{1}{4}(e^4 - 1)$

$\therefore \; u = -1\, e^{-2(0-t)} \dfrac{4}{e^4 - 1} x_{10} = -\dfrac{4}{e^4-1} e^{2t} x_{10}$

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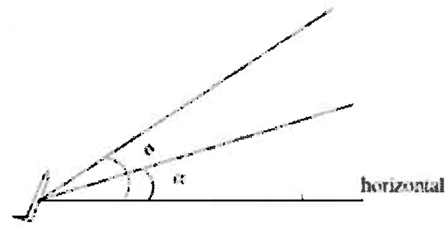
input to steer x_{10}, x_{20} to zero in 1 sec if $\dfrac{x_{10}}{x_{20}} = \dfrac{1}{2}$

5) The linearized dynamics of an airplane are given by

$$\begin{aligned}\dot{\alpha} &= a(\phi - \alpha)\\ \ddot{\phi} &= -\omega^2(\phi - \alpha - bu)\\ \dot{h} &= c\alpha\end{aligned}$$

where α is the flight path angle (positive for ascending, negative for descending), h is the aircraft's altitude, c is the ground speed, ω is the natural frequency of the pitch angle, u is the control input applied by the elevator surfaces, and a, b are constants. Obtain the eigenvalues of the system and investigate if the aircraft can be steered to any given state through the input u.

$$\begin{bmatrix}\dot{\alpha}\\ \dot{\varphi}\\ \ddot{\varphi}\\ \dot{h}\end{bmatrix} = \underbrace{\begin{bmatrix}-a & a & 0 & 0\\ 0 & 0 & 1 & 0\\ +w^2 & -w^2 & 0 & 0\\ c & 0 & 0 & 0\end{bmatrix}}_{A}\begin{bmatrix}\alpha\\ \varphi\\ \dot{\varphi}\\ h\end{bmatrix} + \underbrace{\begin{bmatrix}0\\ 0\\ w^2 b\\ 0\end{bmatrix}}_{B} u$$



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$$\det(\lambda I - A) = \det\left(\begin{bmatrix}\lambda+a & -a & 0 & 0\\ 0 & \lambda & -1 & 0\\ -w^2 & w^2 & \lambda & 0\\ 0 & 0 & 0 & \lambda\end{bmatrix}\right) = \lambda\begin{vmatrix}\lambda+a & -a & 0\\ 0 & \lambda & -1\\ -w^2 & w^2 & \lambda\end{vmatrix}$$

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$$= \lambda\left[(\lambda+a)(\lambda^2+w^2) - w^2(a)\right] = \lambda\left(\lambda(\lambda^2+w^2) + a\lambda^2 + \cancel{aw^2} - \cancel{aw^2}\right)$$

$$= \lambda^2(\lambda^2 + a\lambda + w^2) \longrightarrow \lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_{3,4} = \frac{-a}{2} \pm \frac{\sqrt{a^2-4w^2}}{2}$$

Controllability : $P = \begin{bmatrix}0 & 0 & aw^2b & -a^2w^2b\\ 0 & w^2b & 0 & -w^4b\\ w^2b & 0 & -w^4b & aw^4b\\ 0 & 0 & 0 & caw^2b\end{bmatrix}$

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$$\det(P) = caw^2b\left(\begin{vmatrix}0 & 0 & aw^2b\\ 0 & w^2b & 0\\ w^2b & 0 & -w^4b\end{vmatrix}\right) = caw^2b\left(aw^2b(-w^4b^2)\right)$$

$$= -ca^2w^8b^4 \neq 0$$

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Controllable