School of Mechatronic Systems Engineering
Simon Fraser University
MSE483/782 Midterm Exam

February 23, 2017 (Duration: 2 hours)
Please read the following before signing your name

- The exam is closed-book. A 2-page formula sheet is permitted but must not contain any solved problems. The formula sheet has to be returned with the questions.
- Questions have an equal weight of 20% each. Please clearly specify any assumptions you make and write legibly. You may lose marks if your work is not clear.

Name: Student I.D. Number:

1) Obtain a state-space representation of the following transfer function in the controllable canonical form

$$H(s) = \frac{Y(s)}{U(s)} = \frac{100s}{s^2+100s+9} + 1$$

Show all steps in the derivation.

$$H(s) = \left(\frac{1}{s^2+100s+9}\right)\cdot(100s)+1 = H_1(s)\cdot \underbrace{H_2(s)+1}_{W(s)} = \frac{Y(s)}{U(s)} \Rightarrow Y(s) = H_1(s)U(s)H_2(s)+U(s)$$

$$H_1(s) = \frac{1}{s^2+100s+9} = \frac{W(s)}{U(s)} \longrightarrow \ddot{w} + 100\dot{w} + 9w = u$$

Let $x_1 = w$, $x_2 = \dot{w}$ $\Rightarrow$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \ddot{w} = u - 100\dot{w} - 9w$$
$$= u - 100x_2 - 9x_1$$

$$\therefore Y(s) = W(s)H_2(s) + U(s) = W(s)100s + U(s) \Rightarrow y = 100\dot{w} + u = 100x_2 + u$$

$$\therefore \begin{cases} \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -9 & -100\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u \\ y = \begin{bmatrix}0 & 100\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + 1.u \end{cases}$$

2) Linearize the system below about the origin and obtain the A, B, C, D state-space matrices.

$$\begin{aligned}
\dot{x}_1 &= (-\alpha_1 + \sin(x_2))x_1 + x_2\sin(x_2) + u \\
\dot{x}_2 &= x_1\sin(x_1) + x_2(-\alpha_2 + \sin(x_2)) + u \\
y &= \sin(x_1) + u\cos(x_2)
\end{aligned}$$

$$\left.\begin{aligned}
\sin x_2 &\approx x_2 \\
\cos x_2 &\approx 1 \\
\sin x_1 &\approx x_1 \\
\cos x_1 &\approx 1
\end{aligned}\right\}\ \text{around the origin } x_1, x_2 = 0$$

$$\therefore \left\{\begin{aligned}
\dot{x}_1 &= \left(-\alpha_1 + x_2 - \frac{x_2^3}{3!} + \dots\right)x_1 + x_2\left(x_2 - \frac{x_2^3}{3!} + \dots\right) + u \\
\dot{x}_2 &= x_1\left(x_1 - \frac{x_1^3}{3!} + \dots\right) + x_2\left(-\alpha_2 + x_2 - \frac{x_2^3}{3!} + \dots\right) + u \\
y &= x_1 - \frac{x_1^3}{3!} + \dots + u\left(1 - \frac{x_2^2}{2!} + \dots\right)
\end{aligned}\right.$$

Neglecting order 2 and above terms:

$$\left\{\begin{aligned}
\dot{x}_1 &= -\alpha_1 x_1 + u \\
\dot{x}_2 &= -\alpha_2 x_2 + u \\
y &= x_1 + u
\end{aligned}\right.
\longrightarrow
\left\{\begin{aligned}
\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} &= \overbrace{\begin{bmatrix}-\alpha_1 & 0 \\ 0 & -\alpha_2\end{bmatrix}}^{A}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \overbrace{\begin{bmatrix}1 \\ 1\end{bmatrix}}^{B} u \\
y &= \underbrace{\begin{bmatrix}1 & 0\end{bmatrix}}_{C}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \underbrace{1}_{D} \cdot u
\end{aligned}\right.$$

~~4~~ 2 points for approach

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2 points for each matrix element

3) Convert the following state-space model into a diagonal representation

$$\dot{x} = \begin{bmatrix} 1 & -4 \\ 3 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u(t)$$
$$y = [1 \ \ 0]x$$

Eigenvalues of A $\longrightarrow$

$$A = \begin{bmatrix} 1 & -4 \\ 3 & -6 \end{bmatrix} \longrightarrow \det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda-1 & 4 \\ -3 & \lambda+6 \end{bmatrix}\right) = 0$$

$$(\lambda-1)(\lambda+6) + 12 = 0 \longrightarrow \lambda^2 + 5\lambda + 6 = 0 \longrightarrow \lambda_1 = -3, \ \lambda_2 = -2$$

5

$$\begin{bmatrix} 1 & -4 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = -3 \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} \longrightarrow \begin{cases} v_{11} - 4v_{12} = -3v_{11} \longrightarrow -4v_{12} = -4 \\ 3v_{11} - 6v_{12} = -3v_{12} \longrightarrow 3v_{11} = 3 \end{cases}$$

$$\longrightarrow \boxed{v_1 = \begin{bmatrix} v_{11} \\ v_{11} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ or its multiples}}$$

5 $$\begin{bmatrix} 1 & -4 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = -2 \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} \longrightarrow \begin{cases} v_{21} - 4v_{22} = -2v_{21} \longrightarrow -4v_{22} = -3v_{21} \\ 3v_{21} - 6v_{22} = -2v_{22} \longrightarrow 3v_{21} = 4v_{22} \end{cases}$$

$$\longrightarrow \boxed{v_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \longrightarrow \text{ or its muliples}}$$

2

Let $T = [v_1 \ \ v_2] = \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix} \longrightarrow x = Tz \longrightarrow T\dot{z} = ATz + Bu$, $y = CTz$

2

$$\therefore \begin{cases} \dot{z} = T^{-1}ATz + T^{-1}Bu \\ y = CTz \end{cases} \longrightarrow \begin{cases} \dot{z} = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} z + T^{-1} \begin{bmatrix} 0 \\ -1 \end{bmatrix} u \\ y = [1 \ \ 0] \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix} z \end{cases}$$

~~7~~ 6

2

$$T^{-1} = \frac{1}{3-4} \begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix}^T = \begin{bmatrix} -3 & 4 \\ 1 & -1 \end{bmatrix}$$

$$\therefore \boxed{\dot{z} = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} z + \begin{bmatrix} -4 \\ 1 \end{bmatrix} u, \quad y = [1 \ \ 4] z}$$

3

$8 \times 0.5 = 4$

4) Consider the state space system, $\dot{x} = Ax + Bu$, where $x \in R^n$ and $u \in R^m$. Assuming that A has n distinct eigenvalues, show that the zero-input response of the system can be written in the following form

$$x(t) = \sum_{i=1}^{n} \alpha_i e^{\lambda_i t} v_i$$

where λ_i and v_i are the $i-th$ eigenvalue and eigenvector of A, respectively. — and α_i are some constants

Hint: An arbitrary initial condition can be written as a linear combination of n eigenvectors (if the eigenvectors form a basis, i.e., are linearly independent vectors).

5 Solution: $$x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau$$

Zero input solution $\longrightarrow$ $x(t) = e^{At} x(0)$

$u \equiv 0$

5 $x(0) = \sum_{i=1}^{n} \alpha_i v_i$ where v_i's are eigenvectors of A $\longrightarrow$ Since A has distinct e/v, the eigenvectors are linearly independent $\Rightarrow$ Any vector $x(0)$ can be written in terms of $v_1, \dots, v_n$

$$\therefore \quad x(t) = e^{At} \sum_{i=1}^{n} \alpha_i v_i = \sum_{i=1}^{n} \alpha_i e^{At} v_i$$

10 $$= \sum_{i=1}^{n} \alpha_i \left(I + \frac{At}{1!} + \frac{A^2 t^2}{2!} + \dots \right) v_i$$

$$= \sum_{i=1}^{n} \alpha_i \left(v_i + \frac{A v_i}{1!} t + \frac{A^2 v_i t^2}{2!} + \dots \right)$$

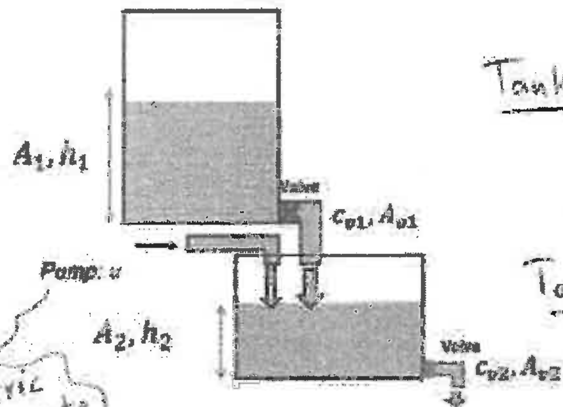
$A A v_i = A \lambda_i v_i = \lambda_i A v_i = \lambda_i^2 v_i$

$$= \sum_{i=1}^{n} \alpha_i \left(v_i + \frac{\lambda_i v_i t}{1!} + \frac{\lambda_i^2 v_i t^2}{2!} + \frac{\lambda_i^3 v_i t^3}{3!} + \dots \right)$$

$$= \sum_{i=1}^{n} \alpha_i \underbrace{\left(1 + \frac{\lambda_i t}{1!} + \frac{\lambda_i^2 t^2}{2!} + \frac{\lambda_i^3 t^3}{3!} + \dots \right)} v_i$$

$$= \sum_{i=1}^{n} \alpha_i e^{\lambda_i t} v_i$$

5) Coupled tanks are common systems in process industries such as petro-chemical, pulp and paper, and water treatment. In these applications, liquids have to be pumped, mixed, stored, and transferred to other tanks. Thus control of the liquid levels is often required by regulating the liquid flows. Assuming that liquid flowing into and out of a tank are given by Q_i and Q_o, respectively, the flow dynamics is given by $Q_i - Q_o = A\frac{dh}{dt}$; where A is the cross sectional area of the tank, and h is the liquid level in the tank, respectively. If the valve is a sharp-edged orifice, its outflow rate is given by $Q_o = C_v a_v \sqrt{2gh}$; where C_v is the discharge coefficient and a_v is the cross sectional area of the valve, respectively, and g is the gravity const. Using the above liquid balance relationship, obtain a state space representation for the system below. Investigate if the system is controllable at a given equilibrium point. If not, indicate how it can be re-designed to make it controllable.



Volumetric flow rate (input)

Tank 1 : $Q_{i1} = 0$

$$Q_{o1} = A_1 \frac{dh_1}{dt} = C_{v1} a_{v1} \sqrt{2gh_1}$$

Tank 2 : $Q_{i2} = Q_{o1} + u$

$$Q_{o2} = C_{v2} a_{v2} \sqrt{2g h_2}$$

Tank 1 : $0 - C_{v1} a_{v1} \sqrt{2gh_1} = A_1 \frac{dh_1}{dt}$

Tank 2 : $Q_{o1} + u - C_{v2} a_{v2} \sqrt{2g h_2} = A_2 \frac{dh_2}{dt}$

$$\therefore \begin{cases} \dot{h}_1 = -\frac{C_{v1} a_{v1}}{A_1}\sqrt{2gh_1} \\ \dot{h}_2 = -\frac{C_{v2} a_{v2}}{A_2}\sqrt{2gh_2} + \frac{C_{v1} a_{v1}}{A_2}\sqrt{2gh_1} + \frac{1}{A_2} u \end{cases}$$

5

Let h_{10}, h_{20}, u_0 be some equilibrium point, i.e.,

$$\begin{cases} \dot{h}_{10} = -\frac{C_{v1} a_{v1}}{A_1}\sqrt{2gh_{10}} \\ \dot{h}_{20} = -\frac{C_{v2} a_{v2}}{A_2}\sqrt{2gh_{20}} + \frac{C_{v1} a_{v1}}{A_2}\sqrt{2gh_{10}} + \frac{1}{A_2} u_0 \end{cases}$$

To simplify notation let

$$\dot{h}_1 = -\alpha_1 h_1^{1/2}$$

$$\dot{h}_2 = -\alpha_2 h_2^{1/2} + \alpha_1 h_1^{1/2} + \beta u$$

where $\alpha_i = C_{vi}\, a_{vi} \sqrt{2g}$, $\beta = 1$

$$\begin{cases} \dot{h}_{10} = -\alpha_1 h_{10}^{1/2} \\ \dot{h}_{20} = -\alpha_2 h_{20}^{1/2} + \alpha_1 h_{10}^{1/2} + \beta u_0 \end{cases} \longrightarrow \text{equilibrium point}$$

Now, let's perform linearization using Taylor's series, i.e.

$$\dot{h}_1 = -\alpha_1 h_{10}^{1/2} - \tfrac{1}{2}\alpha_1 h_{10}^{-1/2}(h_1 - h_{10}) + \cdots$$

$$\dot{h}_2 = -\alpha_2\left(h_{20}^{1/2} + \tfrac{1}{2} h_{20}^{-1/2}(h_2 - h_{20}) + \cdots\right) + \alpha_1\left(h_{10}^{1/2} + \tfrac{1}{2} h_{10}^{-1/2}(h_1 - h_{10}) + \cdots\right) + \beta(u_0 + u - u_0$$

Let $\tilde{x}_1 = h_1 - h_{10}$, $\tilde{x}_2 = h_2 - h_{20}$, $\tilde{u} = u - u_0$. Then,

$$\dot{\tilde{x}}_1 = -\tfrac{1}{2}\alpha_1 h_{10}^{-1/2}\tilde{x}_1$$

$$\dot{\tilde{x}}_2 = -\tfrac{1}{2}\alpha_2 h_{20}^{-1/2}\tilde{x}_2 + \tfrac{1}{2}\alpha_1 h_{10}^{-1/2}\tilde{x}_1 + \beta\tilde{u}$$

$$\begin{bmatrix}\dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2\end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\alpha_1 h_{10}^{-1/2} & 0 \\ \frac{1}{2}\alpha_1 h_{10}^{-1/2} & -\frac{1}{2}\alpha_2 h_{20}^{-1/2} \end{bmatrix}\begin{bmatrix}\tilde{x}_1 \\ \tilde{x}_2\end{bmatrix} + \begin{bmatrix}0 \\ \beta\end{bmatrix}\tilde{u}$$ 5

Controllability: $\mathcal{P} = [B \;\; AB] = \begin{bmatrix} 0 & 0 \\ \beta & -\frac{1}{2}\beta\alpha_2 h_{20}^{-1/2} \end{bmatrix}$

5

$\det(\mathcal{P}) = 0 \longrightarrow$ Not controllable!

Redesign: Pump should be filling up Tank 1 $\longrightarrow$ In current form there is no way h_1 can be controlled

5

$$\begin{matrix} -3 & 4 \\ 4 & -4 \end{matrix} \qquad \begin{matrix} 1 & -4 \\ 3 & -6 \end{matrix} \qquad \begin{matrix} 1 & 1 \\ 1 & 3/4 \end{matrix}$$

$$= \begin{pmatrix} -3 & 4 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} -3 & -2 \\ -3 & -1.5 \end{pmatrix} = \begin{matrix} -3 & 0 \\ 0 & -2 \end{matrix}$$

$$\begin{bmatrix} -3 & 4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -3 & -8 \\ -3 & -6 \end{bmatrix} = \begin{matrix} -3 & 0 \\ 0 & -2 \end{matrix}$$