#1 $A = \begin{bmatrix} \mu & 1 & 0 \\ 0 & \mu & 1 \\ 0 & 0 & \mu \end{bmatrix}$ $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ $C = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}$

Controllability

$$P = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$$

$$= \begin{bmatrix} b_1 & \mu b_1 + b_2 & \mu^2 b_1 + \mu b_2 + \mu b_2 + b_3 \\ b_2 & \mu b_2 + b_3 & \mu^2 b_2 + \mu b_3 + \mu b_3 \\ b_3 & \mu b_3 & \mu^2 b_3 \end{bmatrix}$$

$$= \begin{bmatrix} b_1 & \mu b_1 + b_2 & \mu^2 b_1 + 2\mu b_2 + b_3 \\ b_2 & \mu b_2 + b_3 & \mu^2 b_2 + 2\mu b_3 \\ b_3 & \mu b_3 & \mu^2 b_3 \end{bmatrix}$$

column operations

$$\begin{bmatrix} b_1 & b_2 & 2\mu b_2 + b_3 \\ b_2 & b_3 & 2\mu b_3 \\ b_3 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} b_1 & b_2 & b_3 \\ b_2 & b_3 & 0 \\ b_3 & 0 & 0 \end{bmatrix}$$

$$\det(\cdot) = -b_3^3 \quad \longrightarrow \quad b_3 \neq 0$$

Observability

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_1\mu & c_1 + \mu c_2 & c_2 + \mu c_3 \\ \mu^2 c_1 & 2\mu c_1 + \mu^2 c_2 & c_1 + \mu c_2 + \mu c_2 + \mu^2 c_3 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 & c_2 & c_3 \\ \mu c_1 & \mu c_2 + c_1 & \mu c_3 + c_2 \\ \mu^2 c_1 & \mu^2 c_2 + 2\mu c_1 & \mu^2 c_3 + 2\mu c_2 + c_1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} c_1 & c_2 & c_3 \\ 0 & c_1 & c_2 \\ 0 & 2\mu c_1 & 2\mu c_2 + c_1 \end{bmatrix} \longrightarrow \begin{bmatrix} c_1 & c_2 & c_3 \\ 0 & c_1 & c_2 \\ 0 & 0 & c_1 \end{bmatrix}$$

$$\det(\cdot) = c_1^3 \neq 0 \Longrightarrow c_1 \neq 0$$

#2

(a)
$$\begin{bmatrix}\dot{x}\\ \ddot{x}\\ \dot{y}\\ \ddot{y}\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1 & 0 & 0\\ 9w^2 & 0 & 0 & 2w\\ 0 & 0 & 0 & 1\\ 0 & -2w & -4w^2 & 0\end{bmatrix}}_{A}\begin{bmatrix}x\\ \dot{x}\\ y\\ \dot{y}\end{bmatrix} + \begin{bmatrix}0\\0\\0\\1\end{bmatrix}u.$$

When $u=0$ $\rightarrow$ eig(A) determine stability

$$sI-A = \begin{bmatrix}s & -1 & 0 & 0\\ -9w^2 & s & 0 & -2w\\ 0 & 0 & s & -1\\ 0 & 2w & 4w^2 & s\end{bmatrix}$$

$\det(sI-A)=0$ $\longrightarrow$ expand wrt 1'st or 3rd row:

$$s\begin{vmatrix}s & 0 & -2w\\ 0 & s & -1\\ 2w & 4w^2 & s\end{vmatrix} + 1\begin{vmatrix}-9w^2 & 0 & -2w\\ 0 & s & -1\\ 0 & 4w^2 & s\end{vmatrix}$$

$$= s\left[s(s^2+4w^2) - 2w(-2ws)\right] - 9w^2(s^2+4w^2)$$

$$= s(s^3+4w^2s+4w^2s) - 9w^2s^2 - 36w^4$$

$$= s^4 + 8w^2s^2 - 9w^2s^2 - 36w^4 = s^4 - w^2s^2 - 36w^4 = 0$$

$$\therefore\ s^2 = \frac{w^2 \pm \sqrt{w^4+144w^4}}{2} = \frac{w^2 \pm w^2\sqrt{145}}{2} = w^2\left(\frac{1\pm\sqrt{145}}{2}\right)$$

$$\therefore\ s = \pm w\sqrt{\frac{1+\sqrt{145}}{2}},\ \pm j w\sqrt{\frac{\sqrt{145}-1}{2}}$$

Since one of the e/v is at $w\sqrt{\frac{\sqrt{145}+1}{2}}$ (rhp), the system is unstable!

--- #2 (b)

<u>Pole placement</u> $\quad u = -\begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{bmatrix}$

$$\therefore \; A - BK = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 9\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & -4\omega^2 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 9\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ -k_1 & -k_2-2\omega & -k_3-4\omega^2 & -k_4 \end{bmatrix}$$

$$\therefore \; sI - A + BK = \begin{bmatrix} s & -1 & 0 & 0 \\ -9\omega^2 & s & 0 & -2\omega \\ 0 & 0 & s & -1 \\ k_1 & k_2+2\omega & k_3+4\omega^2 & s+k_4 \end{bmatrix}$$

<u>Expand wrt 1'st row</u> : $\det(sI - A + BK) =$

$$s \begin{vmatrix} s & 0 & -2\omega \\ 0 & s & -1 \\ k_2+2\omega & k_3+4\omega^2 & s+k_4 \end{vmatrix} + 1 \begin{vmatrix} -9\omega^2 & 0 & -2\omega \\ 0 & s & -1 \\ k_1 & k_3+4\omega^2 & s+k_4 \end{vmatrix}$$

$$= s\left[s\left(s^2 + k_4 s + 4\omega^2 + k_3\right) + 2\omega(-s)(k_2+2\omega) \right]$$

$$+ (-9\omega^2)\left(s^2 + k_4 s + k_3 + 4\omega^2\right) + 2\omega(-k_1 s)$$

$$= s\left(s^3 + k_4 s^2 + s\left(\cancel{4\omega^2} + k_3 - 2\omega k_2\right) - \cancel{4\omega^2}\right)$$

$$-9\omega^2\left(s^2 + \underline{k_4 s} + (4\omega^2 + k_3)\right) - \underline{2\omega k_1 s}$$

$$= s^4 + k_4 s^3 + \left(k_3 - 2\omega k_2 - 9\omega^2\right)s^2 - \left(9\omega^2 k_4 + 2\omega k_1\right)s - 9\omega^2(4\omega^2 + k_3)$$

$$\equiv (s+3\omega)(s+4\omega)\left((s+3\omega)^2 + 9\omega^2\right) = \left(s^2 + 7\omega s + 12\omega^2\right)\left(s^2 + 6\omega s + 18\omega^2\right)$$

... 2(b) cont'd

$$\therefore s^4 + k_4 s^3 + (k_3 - 2\omega k_2 - 9\omega^2) s^2 - (9\omega^2 k_4 + 2\omega k_1) s - 9\omega^2(4\omega^2 + k_3)$$

$$\equiv s^4 + 6\omega s^3 + \underline{18\omega^2 s^2} + 7\omega s^3 + \underline{42\omega^2 s^2} + 126\omega^3 s + \underline{12\omega^2 s^2} + 72\omega^3 s + 216\omega^4$$

$$= s^4 + 13\omega s^3 + 72\omega^2 s^2 + 198\omega^3 s + 216\omega^4$$

$$\therefore k_4 = 13\omega$$

$$k_3 - 2\omega k_2 - 9\omega^2 = 72\omega^2$$

$$9\omega^2 k_4 + 2\omega k_1 = 198\omega^3 \Rightarrow 117\omega^3 + 2\omega k_1 = 198\omega^3$$

$$-4\omega^2 - k_3 = 216\omega^4/(9\omega^2)$$

$$\boxed{k_1 = \frac{81}{2}\omega^2} \leftarrow 2\cancel{\omega} k_1 = 81\omega^{\cancel{3}\,2}$$

$$\boxed{k_3 = -4\omega^2 - \frac{216}{9}\omega^2 = -4\omega^2 - 24\omega^2 = -28\omega^2}$$

$$-26\omega^2 - 2\omega k_2 - 9\omega^2 = 72\omega^2 \Rightarrow k_2 = \frac{(-26-9-72)\omega^{\cancel{2}}}{2\cancel{\omega}}$$

$$\boxed{k_2 = -\frac{107}{2}\omega}$$

$$\therefore K = \left[\frac{81}{2}\omega^2 \quad -\frac{107}{2}\omega \quad -28\omega^2 \quad 13\omega\right]$$

#3

$$\begin{cases} L\dfrac{di}{dt} = v - Ri - K_m w \\ J\dfrac{dw}{dt} = K_m i + T_L \end{cases} \longrightarrow \text{Let } z = T_L \text{ be a new state variable.}$$

$\therefore \dot{z} = 0$

Then, we have

$$\begin{bmatrix} \dfrac{di}{dt} \\ \dfrac{dw}{dt} \\ \dfrac{dz}{dt} \end{bmatrix} = \overbrace{\begin{bmatrix} -\frac{R}{L} & -\frac{K_m}{L} & 0 \\ \frac{K_m}{J} & 0 & \frac{1}{J} \\ 0 & 0 & 0 \end{bmatrix}}^{A} \begin{bmatrix} i \\ w \\ z \end{bmatrix} + \overbrace{\begin{bmatrix} 1/L \\ 0 \\ 0 \end{bmatrix}}^{B} v$$

$$y = \overbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}^{C} \begin{bmatrix} i \\ w \\ z \end{bmatrix}$$

Observability: $Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -R/L & -\frac{K_m}{L} & 0 \\ \frac{R^2}{L^2} - \frac{K_m^2}{JL} & \frac{RK_m}{L^2} & -\frac{K_m}{LJ} \end{bmatrix}$

$\det(Q) = -\frac{K_m}{LJ}\left(-\frac{K_m}{L}\right) = \frac{K_m^2}{L^2 J} \neq 0 \longrightarrow$ observable!

A Luenberger observer may be used to estimate the states $\hat{i}$, $\hat{w}$, $\hat{z}$ as follows

$$\begin{bmatrix} \dfrac{d\hat{i}}{dt} \\ \dfrac{d\hat{w}}{dt} \\ \dfrac{d\hat{z}}{dt} \end{bmatrix} = \begin{bmatrix} -R/L & -\frac{K_m}{L} & 0 \\ \frac{K_m}{J} & 0 & 1/J \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{w} \\ \hat{z} \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \\ 0 \end{bmatrix} v + \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} (i - \hat{i})$$

Elements l_1, l_2, l_3 have to be selected depending on desired observer dynamics, i.e., $\text{eig}(A-LC) = \text{eig}\left(\begin{bmatrix} -\frac{R}{L} - l_1 & -\frac{K_m}{L} & 0 \\ \frac{K_m}{J} - l_2 & 0 & \frac{1}{J} \\ -l_3 & 0 & 0 \end{bmatrix}\right)$

#3, continued

$$\det\left(\begin{bmatrix} s+R/L-l_1 & +\frac{K_m}{L} & 0 \\ -\frac{K_m}{J}-l_2 & s & -1/J \\ -l_3 & 0 & s \end{bmatrix}\right) =$$

$$-l_3 \begin{vmatrix} \frac{K_m}{L} & 0 \\ s & -1/J \end{vmatrix} + s \begin{vmatrix} s+R/L-l_1 & K_m/L \\ -\frac{K_m}{J}+l_2 & s \end{vmatrix} = 0$$

$$\therefore\ -l_3\left(-\frac{K_m}{J}\right) + s\left(s^2+\left(\frac{R}{L}-l_1\right)s + \left(\frac{K_m}{J}-l_2\right)\frac{K_m}{L}\right) = 0$$

$$s^3 + \underbrace{\left(\frac{R}{L}-l_1\right)}_{= a_{2,d}} s^2 + \underbrace{\frac{K_m}{L}\left(\frac{K_m}{J}-l_2\right)}_{= a_{1,d}} s + \underbrace{\frac{K_m l_3}{J}}_{= a_{0,d}} = 0$$

$$\therefore \begin{cases} l_1 = \frac{R}{L} - a_{2,d} \\ l_2 = -\frac{L\, a_{1,d}}{K_m} + \frac{K_m}{J} \\ l_3 = \frac{J\, a_{0,d}}{K_m} \end{cases}$$

#4) Separation principle: Observer error dynamics

(a) i.e., eigenvalues and closed-loop eigenvalues when using observer-based state feedback can be allocated separately.

System must be controllable & observable

Proof.

plant
$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

Observer
$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \\ \hat{y} = C\hat{x} \end{cases}$$

state feedback $u = -K\hat{x} + r$

Define observer error as $e = \hat{x} - x$. Then

$$\dot{\hat{x}} - \dot{x} = A\hat{x} + \cancel{Bu} + LC(x - \hat{x}) - Ax - \cancel{Bu}$$

$$\therefore \dot{e} = Ae - LCe = (A - LC)e$$

$$u = -K\hat{x} + r = -K(e + x) + r \longrightarrow \dot{x} = Ax + Bu = Ax - BK(e + x) + r$$

$$\dot{x} = (A - BK)x - BKe + r(t)$$

$$\therefore \begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \underbrace{\begin{bmatrix} A - BK & -BK \\ 0 & A - LC \end{bmatrix}}_{A_c} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} r(t)$$

eig(A_c) consists of eig($A - BK$), eig($A - LC$) which can be indipendently allocated iff the system is controllable & observable!

#4 b

$$\begin{cases} \dot{x} = \overbrace{\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}}^{A} x + \overbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}^{B} u \\ y = \overbrace{\begin{bmatrix} 1 & -1 \end{bmatrix}}^{C} x \end{cases}$$

Controllability: $P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \longrightarrow$ controllable!

Observability: $Q = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \longrightarrow \det(Q) = 1-2 = -1 \longrightarrow$ observable!

State feedback controller $u = -K\hat{x}$, where $\dot{\hat{x}} = A\hat{x} + Bu + L(y-\hat{y})$, $\hat{y} = C\hat{x}$

$$\mathrm{eig}(A-BK) = \mathrm{eig}\left(\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} k_1 & k_2 \end{bmatrix}\right)$$

$$= \mathrm{eig}\left(\begin{bmatrix} -1-k_1 & 1-k_2 \\ 1-k_1 & -k_2 \end{bmatrix}\right)$$

$$|sI - A + BK| = \det\left(\begin{bmatrix} s+1+k_1 & -1+k_2 \\ -1+k_1 & s+k_2 \end{bmatrix}\right)$$

$$= s^2 + (k_1+k_2+1)s + k_2(\cancel{k_1}+1) - (1-k_2-k_1+\cancel{k_1k_2})$$

$$= s^2 + (k_1+k_2+1)s + 2k_2 + k_1 - 1$$

Closed loop eigenvalues are at $-10, -10$, i.e. $(s+10)^2 = s^2 + 20s + 100$

$$\therefore \begin{cases} k_1 + k_2 + 1 = 20 \\ 2k_2 + k_1 - 1 = 100 \end{cases} \longrightarrow \begin{cases} k_1 + k_2 = 19 \\ k_1 + 2k_2 = 101 \end{cases}$$

$$\therefore \boxed{k_2 = 82} \quad \boxed{k_1 = -63}$$

Observer error dynamics $(s+50)^2 = s^2 + 100s + 2500$

$$A - LC = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix}\begin{bmatrix} 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \ell_1 & -\ell_1 \\ \ell_2 & -\ell_2 \end{bmatrix} = \begin{bmatrix} -1-\ell_1 & 1+\ell_1 \\ 1-\ell_2 & \ell_2 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} s+1+\ell_1 & -1-\ell_1 \\ \ell_2-1 & s-\ell_2 \end{bmatrix}\right) = s^2 + (1+\ell_1-\ell_2)s \cancel{-\ell_2} \cancel{+\ell_1\ell_2} + \cancel{\ell_1\ell_2} \cancel{+\ell_2} - \ell_1 - 1$$

$$\equiv s^2 + 100s + 2500 \Longrightarrow \boxed{\ell_1 = -2501, \ \ell_2 = -260}$$

#5

$$u_1 \longrightarrow \boxed{\begin{array}{c} \text{①} \\ \dot{x}_1 = A_1 x_1 + B_1 u_1 \\ y_1 = C_1 x_1 \end{array}} \xrightarrow{y_1 = u_2} \boxed{\begin{array}{c} \text{②} \\ \dot{x}_2 = A_2 x_2 + B_2 u_2 \\ y_2 = C_2 x_2 \end{array}} \longrightarrow y_2$$

$C_1 x_1$

The ~~…~~ cascaded system is not necessarily both controllable and observable.

If there is a pale-zero cancelation between systems ① & ②, then the final system is not minimal and thus uncontrollable/unobservable.