**MSE 426/726 Introduction to Optimal Design**
**Term Test 1**

**Feb 6, 2017**

Name: ____________________ Student Number: ____________________

**Problem 1**: True (T) or False (F); Fill "A" for True and "B" for False on the bubble sheet. (30 marks)

1. The number of equality and inequality constraints could be unlimited in an optimization problem. (False)
2. If a constraint is active for an optimization problem, then the optimum should definitely lie on the constraint boundary. (True)
3. In the steepest descent method, the search direction is in the opposite direction of the gradient. (True)
4. For the three starting points of the quadratic interpolation method, the function value of the middle point should be less than those of the two end points. (True)
5. More constraints tend to increase the value of the objective function at its optimum point. (True)
6. The Newton-Raphson method converges to the optimum when starting at any point in the feasible region. (False)
7. For the Newton-Raphson method, as long as the second derivative of the objective function is larger than or equal to zero, it will find the minimum. (False)
8. The effectiveness of fmincon doesn't depend on the chosen initial point. (False)
9. If f(x) is convex in R, then –f(x) should be concave in R. (TRUE)
10. If $f'(x_0) = 0$ and $f''(x_0) > 0$, then $x_0$ is a global minimum for f(x). (False)
11. The necessary condition for $x_0$ to be a minimum point of f(x) is determined from the Hessian of the objective function. (False)
12. Hessian of a concave function is positive-semidefinite. (False)
13. The large-step univariate search advances an optimal step size along each variable component direction. (True)
14. In all the optimization methods, the gradient information is necessary to find the optimum. (False)
15. The Powell's conjugate direction search method uses the history of previous iterations to build up search directions. (True)

**Problem 2: Multiple-choice questions:**

**Please read the following problem statement and answer questions 16-18.**

A paint company produces interior and exterior paints for buildings from two raw materials with the commercial names M1 and M2. The data of the problem is shown in the following table:

| | Tons of raw material needed to produce one ton of | | Maximum daily availability (tons) | Cost ($1000/ton) |
|---|---|---|---|---|
| | Exterior paint | Interior paint | | |
| M1 | 3 | 5 | 20 | 6 |
| M2 | 6 | 8 | 40 | 3 |
| Price ($1000/ton) | 50 | 70 | | |

The daily demand for interior paint doesn't exceed that for exterior paint by more than 3 tons, according to a market survey. Moreover, the exterior paint has a maximum daily demand of 10 tons. The company owners want to know what would be the optimum mix of interior and exterior paints that could meet the demands and maximize the total daily profit. Consider the following definitions:

$x_1$ : Daily yield of exterior paint in tons

$x_2$ : Daily yield of interior paint in tons

16. Which of the following expressions shows the objective function?

a) $-(50x_1 + 70x_2)$

b) $-(50x_1 + 70x_2) + (6x_1 + 3x_2)$

c) $-(14x_1 + 16x_2)$ (The answer)

d) $-(2x_1 + 28x_2)$

e) None of the above

17. Which of the following is not among the constraints?

a) $3x_1 + 5x_2 - 20 \le 0$

b) $6x_1 + 8x_2 - 40 \le 0$

c) $-x_1 + x_2 - 3 \le 0$

d) $x_1 + x_2 - 10 \le 0$ (The answer)

18. Assume that, by implementing the golden section method on an optimization problem, the following table is obtained for the last iteration. What could be the stopping criterion ($|b_u - b_l| < \varepsilon$) that has resulted in stopping the iterations at iteration#20 and what is the optimum point based on the results?

| Iteration# | $b_l$ | $b_u$ | $x_l$ | $x_u$ | $\lvert b_u - b_l \rvert$ |
|---|---|---|---|---|---|
| 20 | 1.0049 | 1.0083 | 1.0062 | 1.0070 | 0.0034 |

a) $\varepsilon = 0.005, \quad x^* = 1.0049$

b) $\varepsilon = 0.003, \quad x^* = 1.0062$

c) $\varepsilon = 0.005, \quad x^* = 1.0062$ (The answer)

d) $\varepsilon = 0.005, \quad x^* = 1.0083$

19. Which one shows a direction of descent for $f(\vec{x}) = x_1^2 + x_2^2 - 2x_1 - x_2 + 4$ at $\vec{x}_0 = (0,1)$?

a) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ b) $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ c) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (The answer) d) $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$

20. What is the steepest direction of descent for $f(\vec{x}) = x_1^2 + x_2^2 - 2x_1 - x_2 + 4$ at $\vec{x}_0 = (0,1)$?

a) $\begin{bmatrix} 0 \\ -300 \end{bmatrix}$ b) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ c) $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ d) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ (The answer)

21. If the steepest descent direction is $\vec{d} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$ and the starting point is $\vec{x}_0 = (1,1)$, which one shows the function that should be minimized to find the optimum step size if the objective function $f(\vec{x}) = 2x_1^2 + x_2^2 - x_1 - x_2$ is to be minimized by the steepest descent method?

a) $f(\alpha) = 2\alpha^2 + \alpha^2 - \alpha - \alpha$

b) $f(\alpha) = 19\alpha^2 - 10\alpha + 1$ (The answer)

c) $f(\alpha) = 10\alpha^2 - 4\alpha$

d) $f(\alpha) = 51\alpha^2 - 6\alpha$

e) None of the above

22. If $\vec{x}_0 = (0,1)$, $\vec{d} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, and $f(\alpha) = 5\alpha^2 - 2\alpha$, which one shows $\vec{x}_1$?

a) $\vec{x}_1 = (0.2, 1.2)$

b) $\vec{x}_1 = (-0.4, 0.6)$

c) $\vec{x}_1 = (-0.2, 0.8)$ (The answer)

d) $\vec{x}_1 = (0.4, 1.4)$

23. Select the correct Hessian of $f(\vec{x}) = x_2 e^{x_1}$ at point (0,1).

a) $\begin{pmatrix} 0 & e \\ e & 0 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ (The answer)

c) $\begin{pmatrix} e & 1 \\ 1 & e \end{pmatrix}$

d) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

24. Which one shows the type of $[H_0] = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 4 \\ 2 & 4 & 6 \end{pmatrix}$?

a) Positive definite
b) Negative definite
a) Positive-semidefinite (The answer)
c) Indefinite

25. In the Powell's conjugate direction search method, if we have $\vec{x}_0 = (1,1)$, $\vec{x}_1 = (2,1)$, $\vec{x}_2 = (2,5)$, and $\vec{x}_3 = (7,5)$, which one shows the new search direction?

a) $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$ b) $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ (The answer) c) $\begin{bmatrix} 6 \\ 4 \end{bmatrix}$ d) $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

**Problem 3:** The function $f(x) = (x-2)^2 e^x$ should be minimized. The minimum lies somewhere between x=0 and x=4. Answer the following question:

(a) Without solving the optimization problem and based on your engineering insight, where do you think the minimum point and how much do you think the minimum value could be? Provide your answer and the reasons.
(b) Find the relevant equation for finding the solution by the Newton-Raphson method.

Solution:
(a) Both parts of $f(x)$, i.e. $(x-2)^2$ and $e^x$ are nonnegative in the [0,4] interval. The part $(x-2)^2$ is clearly zero at x=2 and positive for all other values of x and, therefore, has a minimum at x=2. The other part, i.e. $e^x$, is always positive (and strictly increasing) for any value of x in the [0,4] interval. Therefore, the minimum of $f(x)$ should occur at x=2, and the minimum value should be zero.
(b) The solution for the relevant Newton-Raphson equation is as follows:

$$f'(x) = 2(x-2)e^x + (x-2)^2 e^x = (2x - 4 + x^2 - 4x + 4)e^x = (x^2 - 2x)e^x$$

$$f''(x) = (2x-2)e^x + (x^2 - 2x)e^x = (x^2 - 2)e^x$$

$$x_{k+1} = x_k - \frac{f'(x)}{f''(x)} = x_k - \frac{(x^2 - 2x)e^x}{(x^2 - 2)e^x} = x_k - \frac{x^2 - 2x}{x^2 - 2}$$

**Problem 4:** Solve the following problem by applying the graphical method. In your solution, you should clearly illustrate the feasible space and the optimum point, e.g. by using arrows and writing next to them.

$$\text{Min } f\left(x_1, x_2\right) = x_1^2 + x_2$$
$$\text{Subject to: } h_1: \quad x_1^2 + x_2^2 = 16$$
$$g_1: \quad x_1 + x_2 \le 2$$
$$g_2: \quad x_1 \ge 0$$

Solution:
The feasible solution space is shown in the following figure according to the constraints. The minimum value which makes $\left\{x_1^2 + x_2 = \text{Constant}\right\}$ curves to be in the feasible solution space is the value associated with the point named "Opt", meaning that the optimum point and value are (0,-4) and -4, respectively.